Fingerprints of Chaos

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Abstract

The asymptotic distance between trajectories d_∞, is studied in detail to characterize the occurrence of chaos. We show that this quantity is quite distinct and complementary to the Lyapunov exponents, and it allows for a quantitave estimate for the folding mechanism which keeps the motion bounded in phase space. We study the behaviour of d_∞ in simple unidimensional maps. Near a critical point d_∞ has a power law dependence on the control parameter. Furthermore, at variance with the Lyapunov exponents, it shows jumps when there are sudden changes on the available phase-space.

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One of the most characteristic feature which is emerging when dealing with nonlinear systems is the appearance of chaotic motion. There has been a considerable amount of work to establish what are the conditions for a nonlinear system (dissipative or conservative) to display chaotic dynamics and what are some suitable quantities to characterize it. The following gives a brief review of the quantities discussed so far in the literature [1]:

i) The Lyapunov exponent (LE), the mean rate of separation between two adjacent trajectories in phase space, is one of the most used measures. A system is chaotic when the trajectories diverge exponentially, i.e. the LE is larger than zero.

ii) The correlation function characterizes the "memory" along one trajectory. It decays quickly to zero in the chaotic regime.

iii) The power spectrum changes from discrete lines to a broad-band noise when chaos sets in.

These quantities can be calculated from the nonlinear equations of motion which describe the system. Often, however the equations of motion are not known but there might be some experimental determination of the time evolution of a physical quantity. There are in the literature some suggestions on how to extract the LE from a time series [2]. However the proposed methods are not unique and depend on some working parameters [3]. Other cases of interest exist where there is no information at all about the time evolution of some quantity and final phase space distributions are only known. For instance in the field of nucleus-nucleus collisions there is an active search for a liquid to gas phase transition at excitation energies of tens MeV/particle or a phase transition to a quark gluon plasma at much higher excitation energies. In these experiments the final momentum distributions (of almost all) particles are known. There is absolutely no information about the time evolution of the system and there is no way to estimate the LE from data. Clearly, it becomes very important to have hold of some clearly defined physical quantity that could define unambiguously the occurrence of a phase transition. It is the purpose of this paper to show that there is a quantity, the asymptotic distance between trajectories, which could be easily estimated in such cases where the quantities (i-iii) cannot be unambiguously determined and which gives for instance as much information as the LE alone. In particular we will describe how the

distance between trajectories, which defines the LE, saturates because of the finiteness of the available phase space. We would like to stress that in the literature [1,2] particular care has been taken to estimate the LE by avoiding such saturation of the trajectories. Here, we are going to do the opposite, i.e. study in detail the saturation properties of trajectories to demonstrate how many physical informations can be obtained from such studies. We will discuss some illustrative examples using one-dimensional maps. Preliminary studies for Hamiltonian systems can be found in [4].

A one-dimensional map can be written as:

$$x_{n+1} = f(x_n, r) \tag{1}$$

where r is a control parameter. Such maps, even though apparently very simple, exhibit most of the chaotic features found in more complex systems.

The appropriate quantity to discuss here is the mean distance between two trajectories separated initially by a very small distance d_0:

$$d_n = \frac{1}{N} \sum_{i=0}^{N} \mid x_n^{(i)} - x'^{(i)}_n \mid \tag{2}$$

and:

$$x_n^{(i)} = f^n(x_0^{(i)}) \tag{3}$$

$$x'^{(i)}_n = f^n(x_0^{(i)} + d_0) \tag{4}$$

In order to avoid fluctuations arising from a particular choice of the starting point for the iterations we average over N couples of trajectories and choose the initial starting points x_0^i randomly from a uniform distribution. After n iterations, for small enough initial separation, the approximate n dependence of the distance d_n is:

$$d_n = e^{n\lambda} d_0 = e^{\lambda} d_{n-1} \equiv \Lambda d_{n-1} \tag{5}$$

where $\Lambda = e^{\lambda}$ and λ is the Lyapunov exponent of the map:

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{i=n} ln \mid f'(x_n) \mid \tag{6}$$

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the prime indicates the derivative of the function $f(x_n)$ respect to x_n [1]. The action of the maps consists generally of two steps: the stretching, which leads to the exponential regime and is characterized quantitatively by the LE and the folding process which keeps the orbit bounded. Therefore when n is sufficiently large, the solution eq.(5) is no longer valid. We can consider eq.(5) as a first order expansion in d_n, and in the hypothesis that: $d_n < 1$ *for any n* , we include a second order correction term in (5):

$$d_{n+1} = \Lambda d_n - \Gamma d_n^2 = F(d_n) \tag{7}$$

Note the analogies to the derivation of the logistic map [5].

Let us define the asymptotic value of the mean distance between two trajectories as:

$$d_\infty = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} d_i \tag{8}$$

The fixed points of (7) are $d_1 = 0$ and:

$$d_2 = d_\infty = \frac{\Lambda - 1.}{\Gamma} \tag{9}$$

Thus, (7) describes the irreversible approach of the system to equilibrium which correspond to the fixed points solution (9). From the stability condition $\mid F'(d_i) \mid < 1$ [1], we find that d_1 is a stable point for $\lambda < 0$, while $d_2 = d_\infty$ is a stable point for $0 < \lambda < ln(3)$. We also stress these particular and interesting cases:

a) $F'(d_\infty) = 0$ gives $\lambda = ln(2)$ which is a superstable point of the map $F(d_n)$. Notice that such value of the LE is obtained in the triangular and logistic maps for a control parameter where the maps become ergodic. *Thus ergodicity of the maps is equivalent to a superstable point of our proposed application describing the evolution of the distance between trajectories.* Also, $d_1 = 0$ becomes a superstable point for $\lambda \to -\infty$.

b)$F'(d_\infty) = -1$ gives $\lambda = ln(3)$ which is the value for which the map $F(d_n)$ has a pitchfork bifurcation. In this case the values of d_∞ at bifurcation are given by the conditions $d_1 = F(d_2)$ and $d_2 = F(d_1)$. This and larger value cases of the LE are outside the purpose of this paper and will be discussed in a future publication [6].

The actual value of Γ can be easily obtained inverting (9):

$$\Gamma = \frac{e^\lambda - 1.}{d_\infty} \tag{10}$$

The entire evolution of the distance between trajectories is given by equation (7). It contains three characteristic quantities, λ, d_∞ and Γ, but only two of them are independent because of the relation (10). To better grasp the meaning of the physical quantities introduced above let us consider the (unbound) map: $x_{n+1} = 2x_n$. It easy to show that for this map it is $\lambda = ln2$, and $\Gamma = 0$. If we impose the condition on the map to have modulus 1 (Bernoulli shift), the LE remains the same while $\Gamma = 1/d_\infty = 3$. Thus the LE is only sensitive to the stretching mechanism while Γ is sensitive to the folding and stretching mechanisms, eq.(10): *its knowledge allows us to distinguish between maps which have the same LE.*

In fig.(1) we plot d_n versus n as obtained numerically for the logistic, triangular and sin maps, for three different initial values of d_0 [7], full lines. We observe in all cases that, after a fast increase, the distance between trajectories saturates. The value of saturation is d_∞ as defined in (8) independent on the initial relative distance d_0. Inserting the values of λ and d_∞, as obtained from eqs.(6) and (8), in eqs. (7) and (10), gives the dot points displayed in fig.(1). The agreement to the numerical results is extremely good for all cases and supporting our hypothesis. *We conclude that in order to have an overall description for the evolution of the mean distance between trajectories, beyond the exponential regime cf. eq.(5), we need two parameters, λ and d_∞, cf.eq.(7).*

At the ergodic point, corresponding to fully developed chaos, we can calculate analytically the LE and d_∞ as mean values over the phase space using the corresponding distribution function [1]. We obtain:

$$\lambda_e = \int_0^1 \rho(x)ln \mid 4 - 8x \mid = ln2 \tag{11}$$

$$d_{\infty e} = \int_0^1 \rho(x)\rho(y) \mid x - y \mid = \frac{4}{\pi^2} \tag{12}$$

for the logistic map. For the triangular map we have $\lambda_e = ln2$, and $d_{\infty e} = 1/3$, the subscript e stands for ergodic. These values are in perfect agreement with the ones used in fig.(1a), and fig.(1c).

For other values of the control parameter it is not possible to calculate analytically λ and d_∞ and we have performed some numerical calculations displayed in fig.(2). Figs.(2a) and (2c) give the LE and d_∞ respectively as obtained from eqs.(6) and (8) for the logistic map. The right column refers to the results of the triangular map. Recall that the LE for the triangular map is given by $\lambda = ln(2r)$. These are some features of particular interest:

i) at particular values of the control parameter, d_∞ has jumps which indicate a change in the dynamical behavior. Notice, for instance, the jump near r=0.7 for the triangular map. This jump,not observed in the LE, is due to the sudden increase of the available phase space, because of band splitting bifurcation, see also fig.(2b);

ii) similarly to the LE, at the transition to regular windows d_∞ drops to zero.

Let us study this last point in more detail for the logistic map. Near the transition point from order to chaos, the LE behaves like

$$\lambda \propto (r - r_\infty)^\beta \qquad ; \qquad \beta = \frac{ln2}{ln\delta} \tag{13}$$

Here r_∞ is the accumulation value of control parameter for the double period bifurcation cascade. This relation can be easily obtained by means of the Renormalization Group Theory (RGT) [1]. Within the same framework we can easily derive d_∞ and obtain [8]:

$$d_\infty \propto (r - r_\infty)^{\nu/2} \qquad ; \qquad \nu/2 = \frac{ln\alpha}{ln\delta} \tag{14}$$

In the inset of fig.(2c), we plot the numerical result for d_∞ vs. r near the critical point. The full line gives the power law, eq.(14), in very good agreement to the numerical values. We remark that the critical exponent for d_∞ depends on both Feigenbaum constants α and δ, while the LE depends on δ only. We have used the symbol ν in analogy to the treatment of second order phase transition. Infact, the counterpart of the distance between trajectories for maps is given by the variance in momentum space for Hamiltonian systems [4]. Near a second order phase transition such a variance is proportional to the inverse of the correlation length which depends on $(T - T_c)^{-\nu}$, where T_c is the critical temperature [9].

To make our analogy to phase transitions more complete recall that at the critical point the correlation function, defined as [1]:

$$C(m, f) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} y_i * y_{i+m} \tag{15}$$

where $y_i = f^i(x_0) - x_{av}$ and $x_{av} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} x_i$, decays with a power law in m:

$$C(m, f_{r_c}) \propto m^{-\eta} \qquad ; \qquad \eta = \frac{2 ln \alpha}{ln 2} \tag{16}$$

which depend now only on α.

All above three critical exponents can be related as follows:

$$\nu = \beta \eta \tag{17}$$

which could be considered valid for the transition to chaos through double period bifurcation. This behaviour is similar for all the maps having period doubling chaotic bifurcation, the critical exponents depending on the values of corresponding α and δ constants.

For completeness in fig.(3) we plot the parameter Γ vs. r for three different maps [7]. We see that this parameter is very large especially near the critical point for the transition from order to chaos, and then it becomes almost constant. The value of the constant is $\frac{1}{d_{\infty e}}$, eq.(12). Thus the LE is to a good approximation proportional to d_∞ especially near the ergodic point. This fact is especially important for equilibrated physical systems since one can deduce the properties of the LE from the variances which are given from thermodynamics [4].

In conclusion, we have shown that a useful quantity to characterize the occurrence of chaos is d_∞, the value at which the distance between two trajectories saturates in phase space. It is complementary to the Lyapunov exponent giving informations about the global features of phase space. It signals also when changes in dynamics are having place which are not always reflected in the behavior of the Lyapunov exponent. Both parameters *are needed* in order to obtain a correct description of the time evolution for the mean distance between trajectories. In analogy to phase transitions, the LE is the order parameter while d_∞ is the inverse of the correlation length. In some experiments both quantities can be measured, often, however, like in nucleus nucleus or cluster cluster collisions, it is not possible to follow the time evolution of the system and asymptotic quantities, like d_∞, can only be detected. The generalized application for the distance between trajectories proposed in this paper,

eq.(7), gives a link between the initial exponential expansion and the final equilibrium stage. We have shown that ergodicity is simply a superstable point of our proposed eq.(7) and naturally gives a Lyapunov exponent $\lambda = ln2$.

REFERENCES

[1] B. Mandelbrot, *The Fractal Geometry of Nature*, Freeman, San Francisco, 1983;

J.L. McCauley, *Chaos Dynamics and Fractals*, Cambridge Nonlinear Science Series 2, Cambridge University Press 1993;

E. Ott, *Chaos In Dynamical Systems*, Cambridge University Press, England, 1993;

R.C. Hilborn, *Chaos and Nonlinear Dynamics*, Oxford University Press, New York, 1994;

H.G.Schuster, *Deterministic Chaos*, VCH,New York,1995;

P.Cvitanovic, *Universality in Chaos*,A.Hilger, Bristol,1989;

J.Froyland, *Chaos and Coherence*, IOP, Bristol, 1992;

M.Tabor, *Chaos and Integrability in Nonlinear Dynamics*, J.Wiley and Sons,New York,1989.

M.Schroeder, *Fractals, Chaos, Power Laws*, W.H.Freeman et C.,1991.

[2] A.Wolf, J.B. Swift, H.L. Swinney and J.A. Vastano "Determining the Lyapunov Exponents From a Time Series", Physica **16D**(1985)285.

[3] A.Casaleggio and S.Braiotta "Estimation of Lyapunov Exponents of ECG Time Series-The Influence of Parameters", Chaos, Sol. and Fract. **8**(1997)1591.

[4] A.Bonasera, V.Latora and M.Ploszajcak "Chaos and Thermodynamics", Ganil preprint (1996) and submitted.

[5] More formally, expanding eq.(1) to second order in δx_n and taking the absolute value gives: $|\delta x_{n+1}| = |f'(x_n)\delta x_n| - 0.5a_n|f''(x_n)\delta^2 x_n|$, where $a_n = \pm 1$ depending on the relative sign of the two terms in the equation above. Averaging over N ensembles and defining: $\Lambda = \frac{1}{N}\frac{\sum_i |f'(x_n,x_0^i)||\delta x_n(x_0^i)|}{d_n}$, and $\Gamma = \frac{1}{2N}\frac{\sum_i a_n|f''(x_n,x_0^i)|\delta x_n^2(x_0^i)}{d_n^2}$, gives eq.(7).

[6] A.Bonasera 'Organizing Chaos', invited report Int.Jour. Mod. Phys.B (in preparation).

[7] The maps used in this paper are [1]:

a) logistic map: $f(x_n) = rx_n(x_n - 1)$, $0 \leq r \leq 4$;

b) triangular map: $f(x_n) = r(1 - 2\mid \frac{1}{2} - x_n \mid)$, $0 \leq r \leq 1$;

c) sine map: $f(x_n) = x_n + r sin(2.\pi x_n)$, $0 \le r \le 0.7326$.

[8] Introducing the doubling operator T [1] and using eqs.(2-4) and (8), we easily get: $d_\infty(Tf) = \alpha d_\infty(f)$. Which can be iterated to: $d_\infty(f) = \alpha^{-n} d_\infty(T^{-n}f)$. Following the same steps as for the derivation of the power law dependence of the LE we obtain eq.(14).
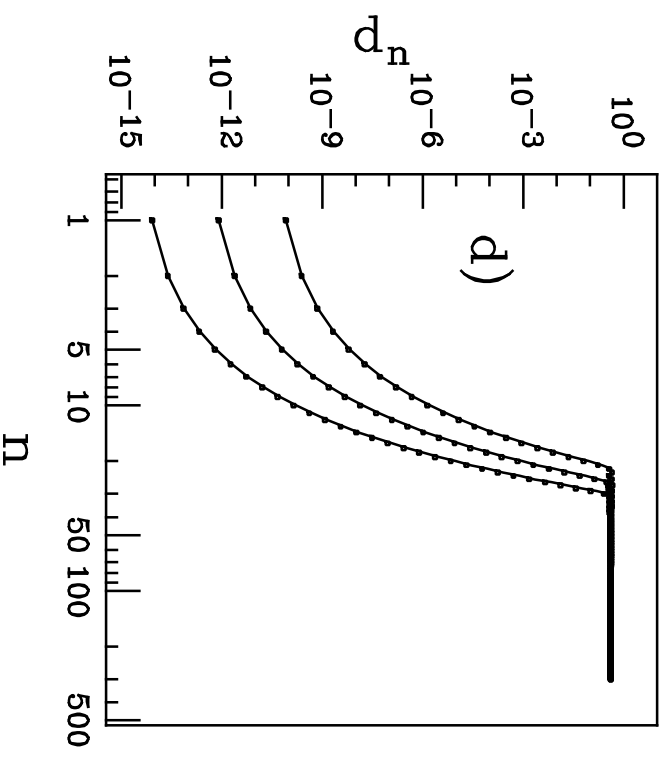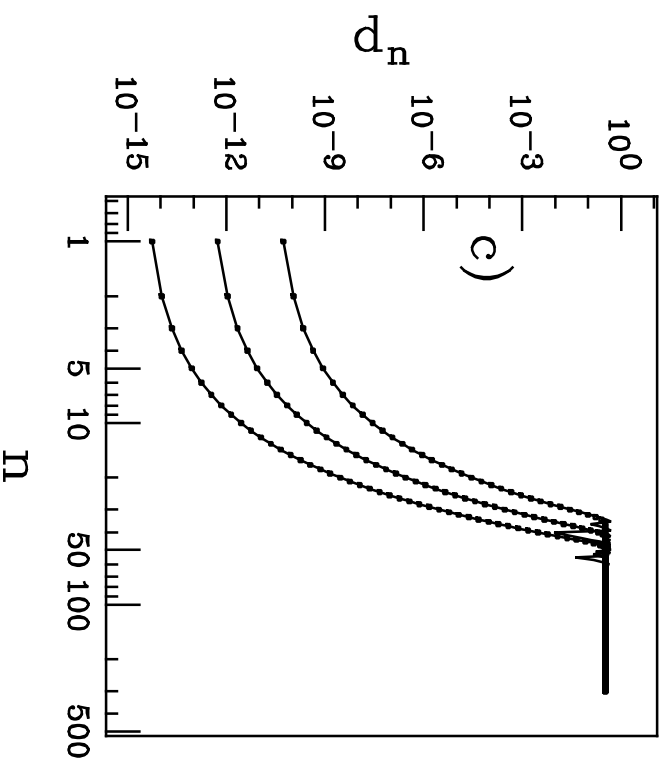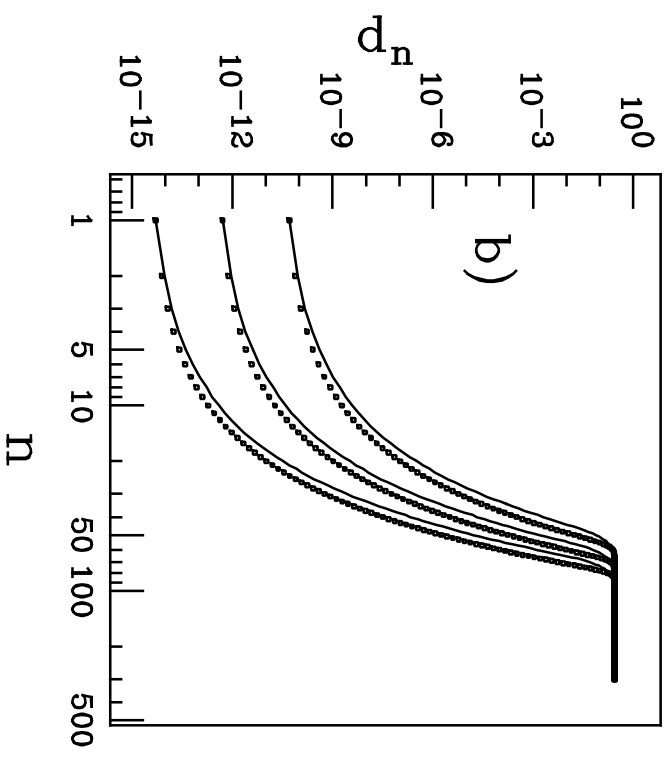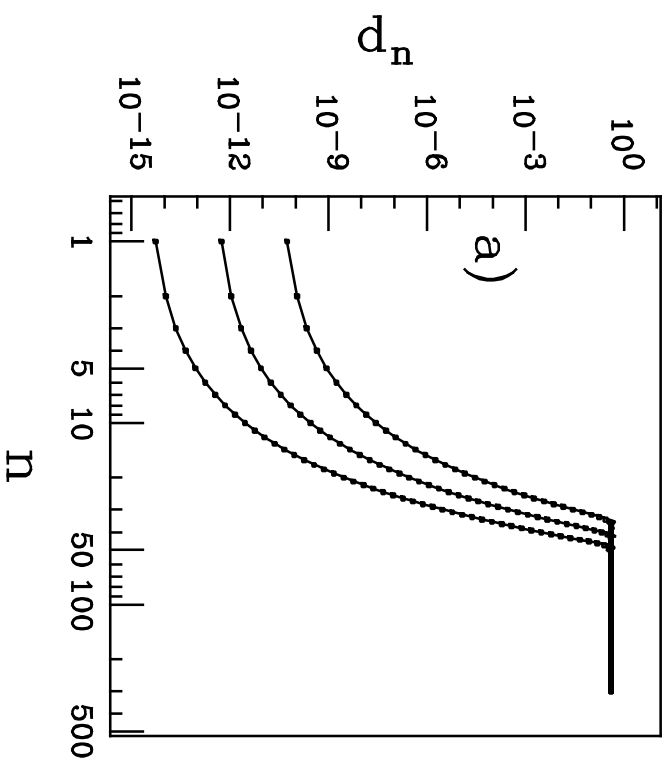
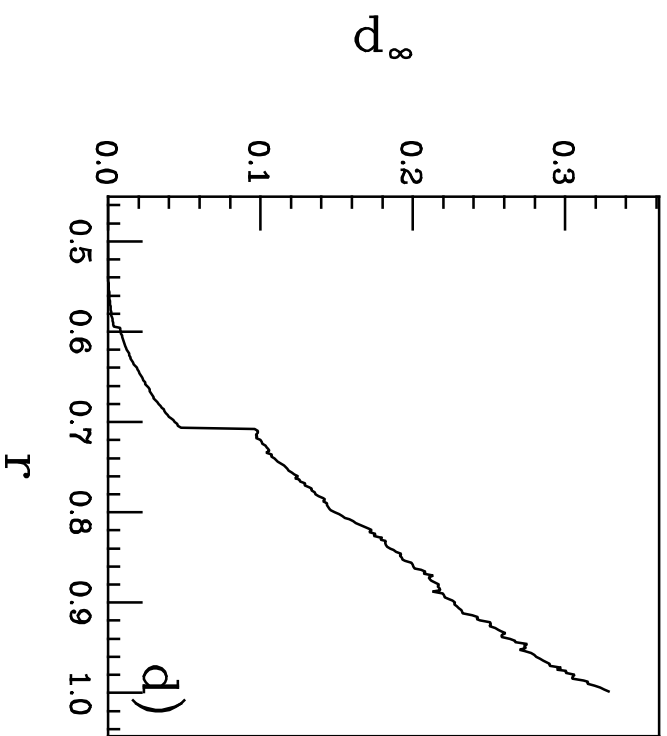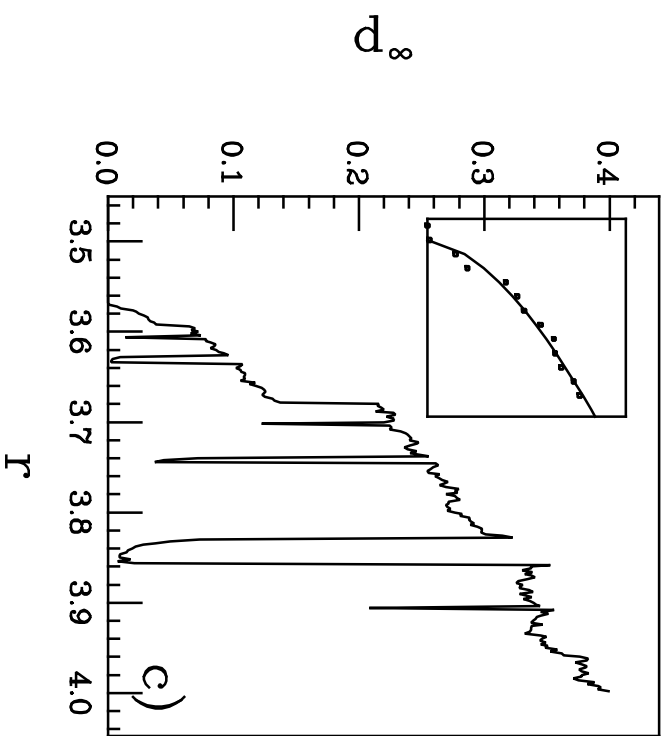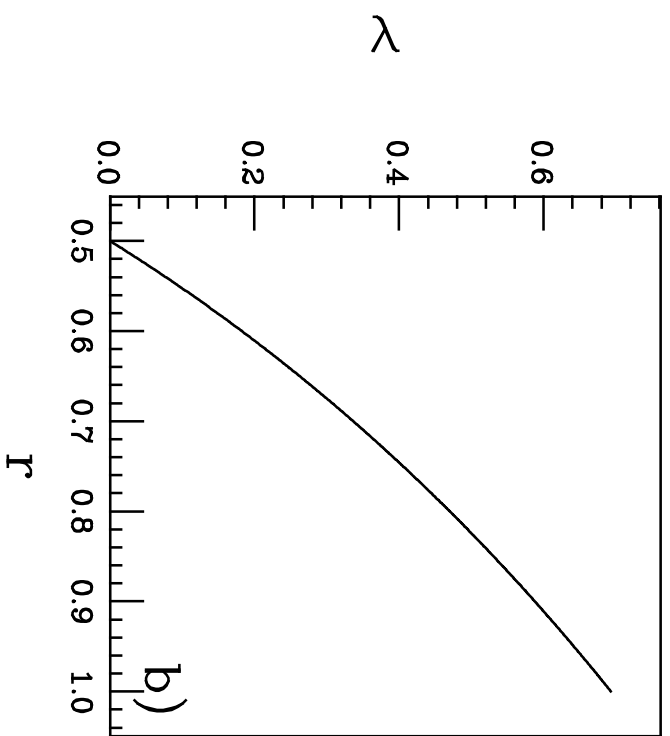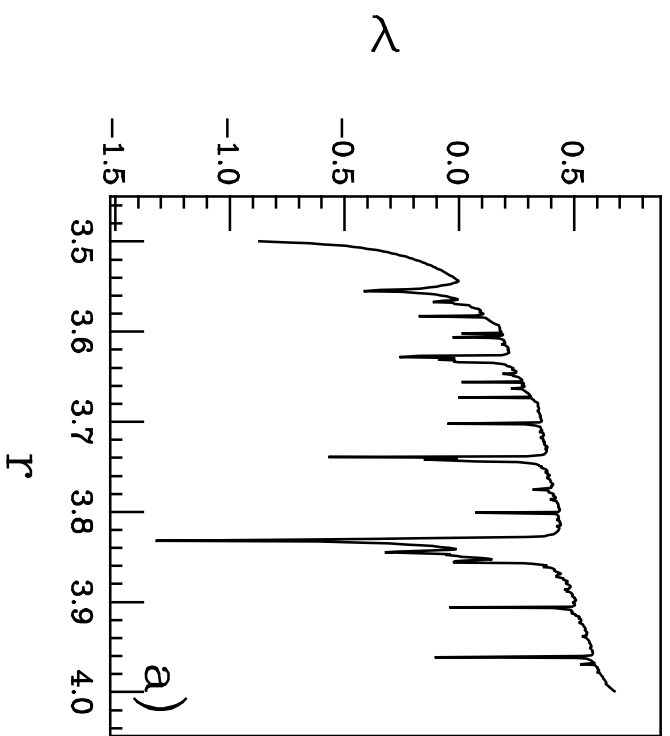[9] E.M. Lifshitz and L.P. Pitaevskii, *Statistical Physics*, part 2, Pergamon Press 1991.

Fig. 1 Distance between trajectories vs. iteration for different maps. (a) Logistic map with the control parameter r=4; (b) the same as (a) but for r=3.771; (c) triangular map at r=1;(d) sine map at r=0.73. The different curves correspond to different starting values of d_0. The dots are obtained from eq.(7).
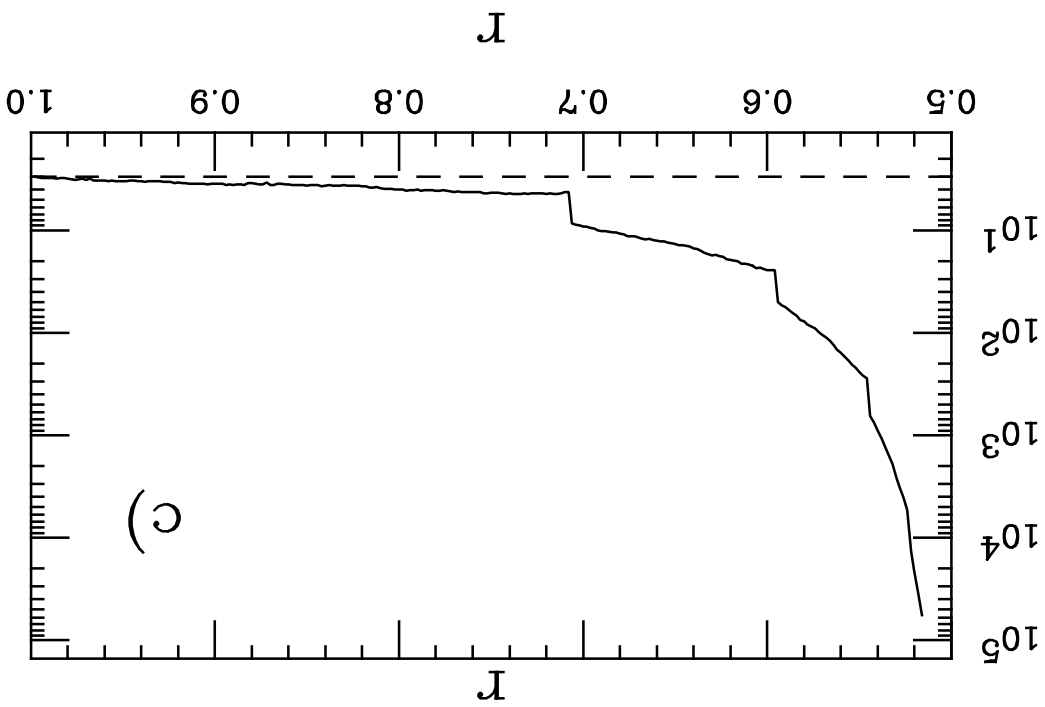
Fig. 2 Lyapunov exponents and asymptotic distances between trajectories vs. control parameter r,for the logistic map (left column) and triangular map (right column). Inset(2c): d_∞ vs. r near the critical point for order to chaos transition, for the logistic map . The full lines give the power law dependence of d_∞ as discussed in the text.

Fig.3 Γ vs. r for the logistic (a), sine(b) and triangular map(c). The dashed line gives the values of $\Gamm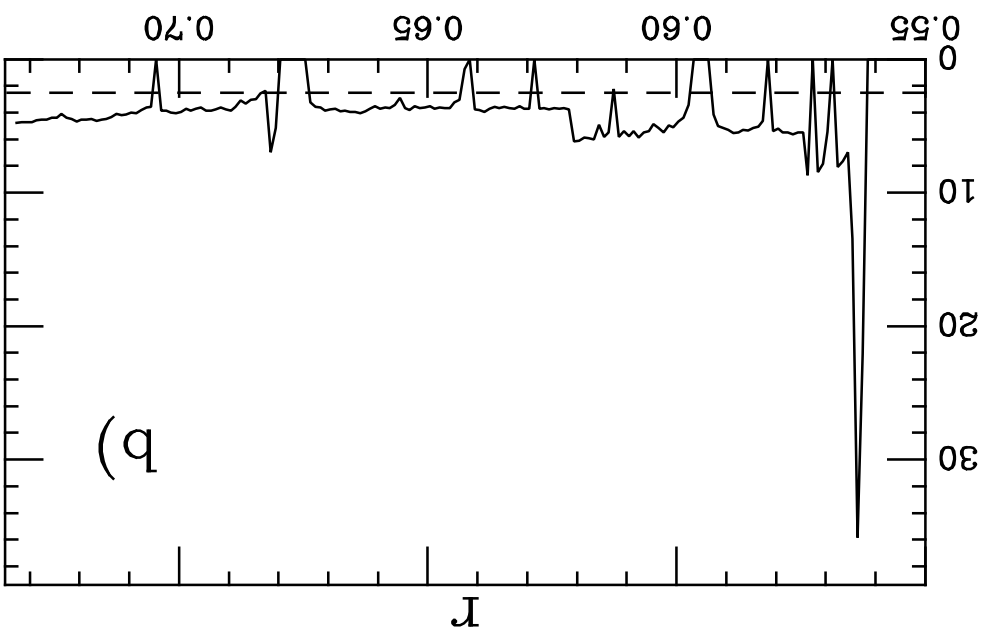a$ when $\lambda = ln(2)$ for the three maps respectively.